A refined polar decomposition for J-unitary operators.

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1 Introduction.

Last years J-symmetric, J-skew-symmetric and J-unitary operators attracted still more and more attention of researches, see, e.g. [1], [2], [3], [4] and references therein. Recall that a conjugation J in a Hilbert space H is an antilinear operator on H such that $J^2x = x$, $x \in H$, and

$$(Jx, Jy)_H = (y, x)_H, \qquad x, y \in H.$$

The conjugation J generates the following bilinear form:

$$[x, y]_J := (x, Jy)_H, \qquad x, y \in H. \tag{1}$$

A linear operator A in H is said to be J-symmetric (J-skew-symmetric) if

$$[Ax, y]_J = [x, Ay]_J, \qquad x, y \in D(A), \tag{2}$$

or, respectively,

$$[Ax, y]_J = -[x, Ay]_J, \qquad x, y \in D(A). \tag{3}$$

A linear operator A in H is said to be J-isometric if

$$[Ax, Ay]_J = [x, y]_J, \qquad x, y \in D(A). \tag{4}$$

A linear operator A in H is called J-self-adjoint (J-skew-self-adjoint, or J-unitary) if

$$A = JA^*J, \tag{5}$$

or

$$A = -JA^*J, \tag{6}$$

or

$$A^{-1} = JA^*J, \tag{7}$$

respectively.

A refined polar decomposition for complex symmetric operators was obtained by Garcia and Putinar in [2]. Using the technique of Garcia and Putinar, an analog for complex skew-symmetric operators was obtained by

Li and Zhou in [4, Lemma 2.3]. In this paper, we shall characterize the components of the polar decomposition for an arbitrary J-unitary operator. This characterization has a quite different structure as that of the above-mentioned decompositions. For the case of a bounded J-unitary operator, a similar decomposition was obtained in [3, Theorem 3.2]. However, in the unbounded case we can not use arguments from [3].
A linear operator A in a Hilbert space H is said to be *J-imaginary* (*J-real*) if $f \in D(A)$ implies $Jf \in D(A)$ and $AJf = -JAf$ (respectively $AJf = JAf$), where J is a conjugation on H. We shall answer a question of the existence of J-imaginary self-adjoint extensions of J-imaginary symmetric operators. This subject is similar to the study of J-real self-adjoint extensions of J-real symmetric operators, see [5]. However, we can not state that a J-imaginary symmetric operator has equal defect numbers. Nevertheless, it is shown that a J-imaginary self-adjoint extension of a J-imaginary symmetric operator exists in a possibly larger Hilbert space.

Notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$, the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively; $\mathbb{R}_e = \mathbb{C}\backslash\mathbb{R}$. By I_d we denote the unit matrix of order d; $d \in \mathbb{N}$. By $\mathfrak{B}(S)$ we mean a set of all Borel subsets of $S \subseteq \mathbb{C}$. If H is a Hilbert space then $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ mean the scalar product and the norm in H, respectively. Indices may be omitted in obvious cases. For a linear operator A in H, we denote by $D(A)$ its domain, by $R(A)$ its range, and A^* means the adjoint operator if it exists. If A is invertible then A^{-1} means its inverse. $\overline{A}$ means the closure of the operator, if the operator is closable. If $A = A^*$, then $\mathcal{R}_z(A) := (A - zE_H)^{-1}$, $z \in \mathbb{R}_e$. For a set $M \subseteq H$ we denote by $\overline{M}$ the closure of M in the norm of H. By E_H we denote the identity operator in H, i.e. $E_H x = x$, $x \in H$. In obvious cases we may omit the index H.

2 Properties of J-unitary operators.

The following proposition accumulates some basic properties of J-unitary operators.

Proposition 1. *Let J be a conjugation on a Hilbert space H, and A be a J-unitary operator in H. Then the following statements are true:*

(i) *A is closed;*

(ii) *A^{-1} is J-unitary;*

(iii) *A^* is J-unitary;*

(iv) A^*A *is* J*-unitary;*

(v) *If* A *is bounded, then* $D(A) = R(A) = H$.

Proof. Let A be a J-unitary operator in a Hilbert space H. By [3, Proposition 2.8] we may write: $A^{-1} = JA^*J = (JAJ)^*$, and therefore A^{-1} and A are closed. Moreover, by [3, Proposition 2.10] we get

$$(A^{-1})^{-1} = (JA^*J)^{-1} = J(A^*)^{-1}J = J(A^{-1})^*J,$$

and therefore A^{-1} is J-unitary. Since

$$(A^*)^{-1} = (A^{-1})^* = JAJ,$$

then A^* is J-unitary. Set $G = A^*A$. The operator G is non-negative and we may write:

$$JG^*J = JA^*AJ = JA^*JJAJ = A^{-1}(A^*)^{-1} = (A^*A)^{-1}.$$

Thus, G is J-unitary.

If A is bounded, then $A^{-1} = JA^*J$ is bounded, and the closeness of A and A^{-1} implies *(v)*.

□

Now we can obtain a refined polar decomposition for a J-unitary operator.

Theorem 1. *Let* J *be a conjugation on a Hilbert space* H*. Then the following assertions hold:*

1) *If* A *is a* J*-unitary operator in* H *then*

$$A = UB, \tag{8}$$

where U *is a unitary* J*-real operator, and* B *is a non-negative self-adjoint* J*-unitary operator;*

2) *If an operator* A *in* H *admits a representation (8) with a unitary* J*-real operator* U*, and a non-negative self-adjoint* J*-unitary operator* B*, then* A *is* J*-unitary.*

Proof. Let A be a J-unitary operator in a Hilbert space H. Set $G = A^*A$, and let $E_G(\delta)$, $\delta \in \mathfrak{B}(\mathbb{R})$ be the spectral measure of G. By Proposition 1 we conclude that G is J-unitary. Let us check that $E(\delta) := JE_G(\delta)J$, $\delta \in \mathfrak{B}(\mathbb{R})$ *is the spectral measure of* G^{-1}. In fact, $E(\delta)$ satisfies conditions $E^2 = E$, $E^* = E$, therefore $E(\delta)$ is a projection operator. The strong σ-additivity of E follows from the strong σ-additivity of E_G and the continuity of J. Moreover, $E(\mathbb{R}) = JE_G(\mathbb{R})J = E_H$. Thus, E is a spectral measure. Denote by T the corresponding to E self-adjoint operator in H. Observe that

$$\mathcal{R}_z(G^{-1}) = (G^{-1} - zE_H)^{-1} = (JG^*J - J\overline{z}J)^{-1}$$

$$= J(G^* - \overline{z}E_H)^{-1}J = J\mathcal{R}_z^*(G)J, \qquad z \in \mathbb{R}_e.$$

For arbitrary $f, g \in H$, $z \in \mathbb{R}_e$, we may write:

$$(\mathcal{R}_z(G^{-1})f, g) = (J\mathcal{R}_z^*(G)Jf, g) = (\mathcal{R}_z(G)Jg, Jf)$$

$$= \int \frac{1}{s-z} d(E_G(s)Jg, Jf) = \int \frac{1}{s-z} d(E(s)f, g) = (\mathcal{R}_z(T)f, g).$$

Therefore $T = G^{-1}$. Notice that

$$\mathcal{R}_z(J|A|J) = (J|A|J - zE_H)^{-1} = (J(|A| - \overline{z}E_H)J)^{-1} = J\mathcal{R}_z^*(|A|)J, \quad z \in \mathbb{R}_e.$$

For arbitrary $f, g \in H$, $z \in \mathbb{R}_e$, we may write:

$$(\mathcal{R}_z(J|A|J)f, g) = (J\mathcal{R}_z^*(|A|)Jf, g) = (\mathcal{R}_z(|A|)Jg, Jf)$$

$$= \int \frac{1}{\sqrt{s} - z} d(E_G Jg, Jf) = \int \frac{1}{\sqrt{s} - z} d(Ef, g) = \left(\int \frac{1}{\sqrt{s} - z} dEf, g \right)$$

$$= (\mathcal{R}_z(\sqrt{G^{-1}})f, g).$$

Therefore

$$J|A|J = \sqrt{G^{-1}}. \tag{9}$$

Let us check that

$$\sqrt{G^{-1}} = \left(\sqrt{G}\right)^{-1}. \tag{10}$$

In fact, using the change of a variable:

$$\lambda = \pi(u) = \begin{cases} \sqrt{u}, & u \geq 0 \\ u, & u < 0 \end{cases},$$

for the spectral measure E_G (see, e.g., [6]) we obtain the spectral measure $E_{\sqrt{G}}$ of $\sqrt{G}$, and we may write:

$$\left(\sqrt{G}\right)^{-1} = \left(\int \sqrt{u} dE_G\right)^{-1} = \left(\int \lambda dE_{\sqrt{G}}\right)^{-1}$$

$$= \int \frac{1}{\lambda} dE_{\sqrt{G}} = \int \frac{1}{\sqrt{u}} dE_G(u). \tag{11}$$

On the other hand, using the change of a variable:

$$\lambda = \widehat{\pi}(s) = \begin{cases} \frac{1}{s}, & s > 0 \\ s, & s \leq 0 \end{cases},$$

for the spectral measure E of G^{-1}, we obtain the spectral measure E_G, and we may write

$$\sqrt{G^{-1}} = \int \sqrt{s} dE(s) = \int \frac{1}{\sqrt{\lambda}} dE_G(\lambda). \tag{12}$$

By (11),(12) we conclude that relation (10) holds.
By (9),(10) we obtain that $J|A|J = |A|^{-1}$. Thus, $B := |A|$ is J-unitary.

Consider the polar decomposition for A: $A = UB$, where U is a unitary operator in H (since $\overline{R(A)} = \overline{R(B)} = H$). Then $A^* = B^*U^*$ (since U is bounded on H) and

$$UB^{-1} = (BU^{-1})^{-1} = (A^*)^{-1} = JAJ = JUJJBJ$$

$$= JUJB^{-1}.$$

Therefore

$$Uh = JUJh, \qquad h \in D(B).$$

By the continuity we conclude that U is J-real.

Let us check assertion 2) of the theorem. For the operator A in this case we may write:

$$JAJ = JUJJBJ = UB^{-1}, \tag{13}$$

$$A^{-1} = B^{-1}U^{-1}. \tag{14}$$

Since U is bounded on H, we may write:

$$JA^*J = (JAJ)^* = (UB^{-1})^* = (B^{-1})^*U^* = B^{-1}U^{-1} = A^{-1}.$$

Therefore A is J-unitary.

□

Corollary 1. *Let J be a conjugation on a Hilbert space H, and A be a J-unitary operator in H. Then operators A^*A and AA^* are unitarily equivalent.*

Proof. In the notations of Theorem 1 we may write: $A^*A = B^2$, and, since U is bounded, $AA^* = UB(UB)^* = UBB^*U^* = UBBU^{-1} = UA^*AU^{-1}$. Observe that we only used that U is unitary in the polar decomposition of A. □

As it was noticed in [2], for the unilateral shift A the operators A^*A and AA^* are not unitarily equivalent. Thus, the unilateral shift is not J-unitary.

Example 1. *(An unbounded J-unitary operator) Let*

$$A_0 := A_0(\beta) := \begin{pmatrix} 0 & \beta i \\ -\beta i & 0 \end{pmatrix}, \qquad \beta \in (-1,1).$$

Observe that

$$(A_0(\beta) \pm I_2)^{-1} = \frac{1}{1-\beta^2} \begin{pmatrix} \pm 1 & -\beta i \\ \beta i & \pm 1 \end{pmatrix}.$$

Let $H = \bigoplus_{k=1}^{\infty} H_k$, where $H_k = \mathbb{C}^2$ is the space of 2-dimensional complex vectors, and $A = \bigoplus_{k=1}^{\infty} A_0\left(1 - \frac{1}{k}\right)$. For an element of H of the form $h = (h_j)_{j=1}^{\infty}$, $h_j = \begin{pmatrix} h_{j,1} \\ h_{j,2} \end{pmatrix} \in H_j$, we set $Jh = (\mathcal{J}h_j)_{j=1}^{\infty}$, where $\mathcal{J}h_j = \begin{pmatrix} \overline{h_{j,1}} \\ \overline{h_{j,2}} \end{pmatrix}$. Observe that J is a conjugation on H. It is straightforward to check that A is a bounded self-adjoint, J-skew-self-adjoint operator on H, and there exist $(E_H \pm A)^{-1}$. Let $e_{k,1}$ be an element of H of the form $(h_j)_{j=1}^{\infty}$, $h_j \in H_j$, where $h_j = \delta_{j,k} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $k \in \mathbb{N}$. Observe that

$$(E_H + A)^{-1} e_{k,1} = \left(\delta_{j,k} \frac{1}{1-(1-\frac{1}{k})^2} \begin{pmatrix} 1 \\ (1-\frac{1}{k})\, i \end{pmatrix} \right)_{j=1}^{\infty}.$$

Since

$$\left((E_H + A)^{-1} e_{k,1}, e_{k,1}\right)_H = \frac{1}{1 - \left(1-\frac{1}{k}\right)^2} \to \infty,$$

as $k \to \infty$, then $(E_H + A)^{-1}$ is unbounded. Consider the following operator:

$$V = (A + E_H)(A - E_H)^{-1} = E_H + 2(A - E_H)^{-1}. \tag{15}$$

Transformation (15), which connects some J-skew-symmetric and J-isometric operators, was studied by Kamerina in [7]. Observe that

$$V = \int \frac{\lambda+1}{\lambda-1} dE_A(\lambda),$$

where $E_A(\lambda)$ is the spectral measure of A. Thus, V is self-adjoint, and we may write:

$$JV^*J = JVJ = E_H + 2J(A-E_H)^{-1}J = E_H - 2(A+E_H)^{-1}$$
$$= (A-E_H)(A+E_H)^{-1} = V^{-1}.$$

Thus, V is a J-unitary operator. Therefore V^{-1} is an unbounded J-unitary operator.

Unitary J-real operators, which appear in the refined polar decomposition (8), also play an important role in the question of an extension of J-imaginary symmetric operators to J-imaginary self-adjoint operators.

Theorem 2. *Let J be a conjugation on a Hilbert space H. Let A be a closed J-imaginary symmetric operator in H, $\overline{D(A)} = H$. Then there exists a J-imaginary self-adjoint operator $\widetilde{A} \supseteq A$ in a Hilbert space $\widetilde{H} \supseteq H$ (with an extension of J). If the defect numbers of A are equal, then there exists a J-imaginary self-adjoint operator $\widehat{A} \supseteq A$ in H.*

Proof. At first, suppose that the defect numbers of A are equal. Consider Cayley's transformation of A:

$$U_z = U_z(A) = (A-\overline{z}E_H)(A-zE_H)^{-1} = E_H + (z-\overline{z})(A-zE_H)^{-1}, \quad z \in \mathbb{C}.$$

Observe that

$$J\mathcal{M}_z(A) = \mathcal{M}_{-\overline{z}}(A), \qquad z \in \mathbb{C},$$

where $\mathcal{M}_\lambda(A) := (A-\lambda E_H)D(A)$, $\lambda \in \mathbb{C}$. In particular, we see that

$$J\mathcal{M}_{\pm i}(A) = \mathcal{M}_{\pm i}(A). \tag{16}$$

Then

$$J\mathcal{N}_{\pm i}(A) = \mathcal{N}_{\pm i}(A), \tag{17}$$

where $\mathcal{N}_\lambda(A) := H \ominus \mathcal{M}_\lambda(A)$, $\lambda \in \mathbb{C}$.

Let W be an arbitrary linear J-real isometric operator, which maps $\mathcal{N}_i(A)$ onto $\mathcal{N}_{-i}(A)$. In particular, if $\mathfrak{A}_\pm = \{f_k^\pm\}_{k=0}^\tau$, $0 \le \tau \le +\infty$, is an orthonormal basis in $\mathcal{N}_{\pm i}(A)$, corresponding to J (i.e. $Jf_k^{\pm} = f_k^{\pm}$), then we may set

$$W \sum_{k=0}^{\tau} \alpha_k f_k^{+} = \sum_{k=0}^{\tau} \alpha_k f_k^{-}, \qquad \alpha_k \in \mathbb{C}.$$

Then $V := U_i \oplus W$ is a J-real unitary operator in H. Observe that $\widetilde{A} := iE_H + 2i(V - E_H)^{-1} \supseteq A$, is self-adjoint and J-imaginary.

In the case of unequal defect numbers, we may consider an operator $\mathcal{A} := A \oplus (-A)$ in a Hilbert space $\mathcal{H} := H \oplus H$ with a conjugation $\mathcal{J} = J \oplus J$. The operator $\mathcal{A}$ is closed symmetric, $\mathcal{J}$-imaginary, $\overline{D(\mathcal{A})} = \mathcal{H}$, and it has equal defect numbers. Thus, we may apply to $\mathcal{A}$ the already proved part. $\square$

Example 2. *(A J-imaginary symmetric operator) Consider the usual space $H = l_2$ of square summable sequences of complex numbers $h = \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \end{pmatrix}$. A conjugation J will be the following one: $Jh = \begin{pmatrix} \overline{h_0} \\ \overline{h_1} \\ \overline{h_2} \\ \vdots \end{pmatrix}$. An operator A we shall define on a set of all finite vectors $\mathcal{F}$ (i.e. vectors which components are zeros except for a finite number) by the following matrix multiplication:*

$$Ah = i \begin{pmatrix} 0 & \alpha_0 & 0 & 0 & \dots \\ -\alpha_0 & 0 & \alpha_1 & 0 & \dots \\ 0 & -\alpha_1 & 0 & \alpha_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} h.$$

It is straightforward to check that A is symmetric and J-imaginary. Observe that $\overline{A}$ is J-imaginary, as well. Applying Theorem 2 to the operator $\overline{A}$ we conclude that the operator A has a self-adjoint J-imaginary extension in a Hilbert space $\widetilde{H} \supseteq H$.

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In this paper, we shall characterize the components of the polar decomposition for an arbitrary J-unitary operator in a Hilbert space. This characterization has a quite different structure as that for complex symmetric and complex skew-symmetric operators. It is also shown that for a J-imaginary closed symmetric operator in a Hilbert space there exists a J-imaginary self-adjoint extension in a possibly larger Hilbert space (a linear operator A in a Hilbert space H is said to be J-imaginary if $f \in D(A)$ implies $Jf \in D(A)$ and $AJf = -JAf$, where J is a conjugation on H).